General Topology and its Applications 2 (1972) 349–362. North-Holland Publishing Company

A TOPOLOGICAL VIEW OF P-SPACES *

Arvind K. MISRA

Mathematisch Instituut, Roetersstraat 15, Amsterdam, The Netherlands
Mathematics Department, Indian Institute of Technology, Kanpur, Kanpur–16, India

Received 4 November 1971

Abstract: A T_1 space in which the intersection of any countably many open sets is open, is called a P-space. In the present paper some basic topological properties of these spaces are studied. Three main results are a characterization of regular paracompact P-spaces, a product theorem for realcompact P-spaces and an example of a Hausdorff connected P-space.

AMS Subj. Class.: Primary 54G10, 54D20; Secondary 54D05, 54D15, 54D60, 54B90

1. Introduction

The term P-space was used as an abbreviation for pseudo-discrete space by Gillman and Henriksen in [9] for a completely regular Hausdorff space X on which every continuous real-valued function is constant on some neighborhood of each point of X. In a P-space the intersection of any countably many open sets is again an open set.[1] We shall take this latter condition with the T_1-axiom as the defining condition of a P-space (not necessarily completely regular). The Tychonoff P-spaces are important in the study of the ring of all continuous real-valued functions

* Most of the results included here are part of the author's thesis submitted to I.I.T. Kanpur in June 1970. Financial assistance was provided by C.S.I.R., New Delhi, during the research.

[1] Sikorski in [17] and Cohen and Goffman in [6] discussed T_1 spaces in which the intersection of any collection of at most $\aleph_\alpha$ open sets is again an open set. These were called ω_α-additive spaces in the former. Suitably weakened concepts of separability, compactness, completeness etc. were defined for ω_α-additive spaces. Whenever needed, we shall refer to these spaces as P_α-spaces, the P_0-spaces being precisely the P-spaces. The countable cardinal being the first infinite cardinal, the class of P-spaces is naturally more closely related to the topological properties known to be of interest than are its subclasses of P_α-spaces, $\alpha \geqslant 1$. Moreover, once the behaviour of P-spaces is known, most of the results about P_α-spaces can be easily developed.

on a Tychonoff space and a number of results about them can be found in [9; 10].[2] P-spaces are also naturally introduced in analysis through Baire functions (see [14]).

The aim of the present paper is to provide a systematic account of the basic properties of P-spaces in order to throw more light on the effects of the closure of the topology of a P-space under countable intersections as against finite intersections only, pointing both the parallelisms and vicissitudes. This, besides providing some interesting examples and answers to basic questions, would also provide the fundamental material for a subsequent paper on the coreflective functor P from the category of all T_1 spaces to the subcategory of P-spaces. Three major results of the paper are a characterization of regular paracompact P-spaces, a product theorem for realcompact P-spaces and an example of a Hausdorff connected P-space.

All spaces throughout the following discussion are at least T_1.

2. The P-spaces

A P-*topology* on a non-empty set X is a topology such that the intersection of any countably many open sets is open. A P-*space* (X,τ) (or simply X) is a non-empty set X equipped with a P-topology τ. In a topological space X a point x is called a P-*point* if the intersection of any countably many neighborhoods of x is again a neighborhood.

Clearly a space X is a P-space if and only if every point is a P-point. The following theorem provides an external characterization of P-spaces.

Theorem 2.1. A topological space X is a P-space if and only if for any Lindelöf space Y the projection $\mathrm{pr}_x : X \times Y \to Y$ is a closed map.

Proof. The necessity part follows exactly on the lines of a similar result involving product with a compact space. The sufficiency of the condition is a consequence of a result of Hanai in [11, Theorem 4] which states that every non-P-space is such that whenever the projection of $X \times Y$ on X is closed, Y has to be countably compact. Therefore, if X is a non-P-

[2] In [1;4;9] interest in P-spaces is not really topological but is mainly in the ring $C(X)$ of continuous real-valued functions on a Tychonoff P-space X. In [9], $T_{3\frac{1}{2}}$ P-spaces were characterized as spaces X for which every prime ideal in $C(X)$ is maximal. Several other equivalent conditions were also obtained. In [1;4], for a P-space X, characterizations of $C(X)$ as a vector lattice and an l-ring were given. The author is thankful to Prof. P. Meyer for the references [1,4].

space, the projection from $X \times R$ on X is not closed but the real line R is a Lindelöf space.

A few examples may be in order at this place.

Every discrete space is certainly a P-space. The simplest examples of non-discrete P-spaces are provided by spaces X of countable cardinal in which every point is isolated except for one with neighborhoods having countable complements. One can also have uncountable sets with P-topology in which every point has neighborhoods with countable complements. The last mentioned spaces have no isolated points but poor separation properties. Interesting examples of P-spaces with no isolated points are provided by η_1-sets (see [10, Exercises 13O, 13P]) with the interval topology.

The following proposition is a simple consequence of the fact that the inverse of a function preserves intersections.

Proposition 2.2. All subspaces, disjoint topological sums, finite products and T_1 quotients of P-spaces are P-spaces.

No infinite product of P-spaces with more than one point is a P-space as every such product contains a copy of 2^ω, and in 2^ω each point is a non-isolated G_δ-set. In fact, given a family $\{X_\gamma \mid \gamma \in \Gamma\}$ of P-spaces a right P-topology on ΠX_γ will be the smallest P-topology which makes all the projections continuous. The product set with this P-topology will be called the P-product of the family $\{X_\gamma \mid \gamma \in \Gamma\}$ and denoted by $\mathrm{P}\{X_\gamma \mid \gamma \in \Gamma\}$. Let a family S of subsets of a P-space (X, τ) be called a P-subbase if S_δ, the family of all countable intersections of members of S, is a base for τ. Then if for each γ, S_γ is a subbase for the topology of X_γ, the family $\cup_\gamma \mathrm{pr}_\gamma^{-1}[S_\gamma]$ is a P-subbase for the P-product topology. The following proposition essentially states that the P-product is really the categorical product in the category of P-spaces. The proof is straightforward.

Proposition 2.3. A function f from a P-space to a P-product $Y = \mathrm{P}\{X_s \mid s \in S\}$ of P-spaces X_s, $s \in S$, is continuous if and only if its composition with each projection map $\mathrm{pr}_s \colon Y \to X_s$ is continuous.

3. Separation axioms

The first thing that can be easily decided is that the T_i-axioms, $i = 1, 2, 3, 4, 5$, maintain their identity in the class of P-spaces. Non-

Hausdorff P-spaces are legion and the P-topology obtained by extending the topology of a T_3 P-space by adding a non-open dense subset is a T_2 P-space which is not T_3 (see [3, Exercise I.8.20]). If $P(\omega_i + 1)$ is the subspace of the ordinal $\omega_i + 1$ obtained by deleting all those points to which a sequence converges then $P(\omega_i + 1)$ is a P-space (see [10, Exercise 5O]) and the subspace of $P(\omega_1 + 1) \times P(\omega_2 + 1)$ obtained by deleting the point (ω_1, ω_2) is a T_3 space which is not T_4. The proof is on the same lines as for a similar subspace of the Tychonoff plank (see [10, Exercise 8J]). The space $P(\omega_1 + 1) \times P(\omega_2 + 1)$ is a nice T_4 P-space but its subspace described above is not normal. The following example, patterned on one in [20] is also interesting in this connection.

Example 3.1. Let E_0 be the union of disjoint sets $\{a, b\}$, $\{a_{\alpha\beta} \mid 0 \leq \alpha, \beta < \omega_1\}$ $\{b_{\alpha\beta} \mid 0 \leq \alpha, \beta < \omega_1\}$ and $\{c_\gamma \mid 0 \leq \gamma < \omega_1\}$. Let the basic neighborhoods of various points be as follows: all the points $a_{\alpha\beta}$ and $b_{\alpha\beta}$, $0 \leq \alpha, \beta < \omega_1$ are isolated; for each fixed γ, a typical basic neighborhood of the point c_γ contains the points $a_{\gamma\beta}$ and $b_{\gamma\beta}$ for all but countably many indices β, $0 \leq \beta < \omega_1$; a typical basic neighborhood of a (respectively b) contains, for every α greater than some ordinal $\delta < \omega_1$, all but countably many points $a_{\alpha\beta}$ (respectively $b_{\alpha\beta}$). The space E_0 with the topology generated by the above defined neighborhood bases is a Hausdorff semi-regular P-space. As every closed neighborhood of each of a and b contains all but countably many points c_γ, E_0 is not a Urysohn space.

The subspace of E_0 obtained by deleting all points $b_{\alpha\beta}$, $0 \leq \alpha, \beta < \omega_1$ and b is a Urysohn P-space which is not semi-regular.

Proposition 3.2. In a P-space, regularity and complete-regularity are equivalent.

Proof. Let X be a regular P-space and F a closed set in X. If x is not in F, then $V_0 = X \setminus F$ is a neighborhood of x and by regularity there is a descending sequence $\{V_n\}$, $n \in \mathrm{N}$, of neighborhoods of x such that $\bar{V}_i \subset V_{i-1}$ for each $i \in \mathrm{N}$. Now the intersection of all V_i's is a clopen neighborhood of x disjoint from F, hence its characteristic function is a continuous real-valued function separating x and F.

The above proof shows that a regular P-space has a clopen base. Also since zero-sets are G_δ's these are precisely the clopen sets and hence (see [10, 6.9 (c)]) the Stone-Čech compactification of a regular P-space also has a clopen base, that is:

Corollary 3.3. A regular P-space is strongly zero-dimensional.

4. Compactness properties

Hausdorff spaces satisfying various compactness properties, sequential spaces and k-spaces form important classes of topological spaces. For P-spaces, as we shall see below, most of these properties are a little too strong and only discrete ones can have them. Nevertheless, as pointed out by Sikorski in [18] suitably weakened compactness properties give, in most respects, equally interesting subclasses of P-spaces. This suitable weakening has quite a discernible pattern and leads to very many easily verifiable parallel results for P-spaces. In this section we shall note down a few of these which we think are important and form useful information. Some important exceptions, which are not amenable to such analogising, for example the Tychonoff theorem and compactifications, are also discussed.

Proposition 4.1. A P-space with any of the following properties is discrete, in fact finite except in case (d):

(a) countable compactness;
(b) sequential compactness;
(c) compactness;
(d) local compactness;
(e) $T_{3½}$ and pseudocompactness;
(f) $\mathfrak{m}$-compactness.

Proof. The properties (b), (c) and (f) are stronger than (a) and in a countably compact space every infinite subset has a limit point, whereas in a P-space every countable subset, being an F_σ-set, is closed and discrete. Therefore a P-space with any of the properties (a), (b), (c) and (f) must be finite and hence discrete.

(d) A compact neighborhood of each point in a locally-compact P-space is finite and discrete by (c) and Proposition 2.7. Therefore each point is isolated.

(e) Tychonoff P-spaces X are characterized by purely algebraic properties of the ring $C(X)$ of real-valued continuous functions on X (see [10]). Now if X is a pseudocompact P-space and βX its Stone-Čech compactification then $C(\beta X)$ is isomorphic to $C(X)$ which implies that βX is a P-space and hence, by (c), that X is finite and hence discrete.

Every sequential space is a k-space and every k-space is the natural quotient of the disjoint sum of its compact subspaces. Therefore, by Propositions 2.1 and 4.1 it follows that any P-space which is either a k-space or a sequential space has to be discrete.

In the next proposition we note down several results which bring out the fact that the Lindelöf property is almost as useful and important for P-spaces as compactness is for general spaces.

Proposition 4.2[3] (a). A Lindelöf subspace of a Hausdorff P-space is closed.

(b). A Hausdorff Lindelöf P-space is normal.

(c). If A is a Lindelöf subset of a regular P-space then for each open neighborhood U of A, there is an open neighborhood V of A whose closure is contained in U.

(d). For any initial ordinal ω_α which is not cofinal with ω_0, the intersection of a descending ω_α-sequence of non-empty Lindelöf subsets of a Hausdorff P-space is itself non-empty and Lindelöf and the sequence is eventually contained in every neighborhood of the intersection.

(e). Every continuous function from a Lindelöf T_2 space to a T_2 P-space is closed and hence every such continuous bijection is a homeomorphism.

(f). On a set, a Lindelöf, Hausdorff P-topology is a maximal Lindelöf topology and a minimal Hausdorff P-topology.

(g). If $f: X \to Y$ is a closed, continuous function from a P-space X onto a P-space Y such that inverse images of points are Lindelöf (i.e., a P-proper map, say) then X is Lindelöf or para-Lindelöf whenever Y is so.

(h). The product of two Lindelöf P-spaces is a Lindelöf P-space.

Proof. A very standard line of argument proves (a), (b) and (c). As the Lindelöf property is preserved under continuous maps and is closed hereditary, (e) follows from (a).

(d). Let $\mathcal{F} = (F_\mu)$ be an ω_α-sequence of the said type. Then by (a) F_0 is closed and $\mathcal{F}$ is a family of closed subsets of F_0 with the countable intersection property and hence $\mathcal{F}$ must have a non-empty intersection. This intersection, being closed in the Lindelöf subspace F_0, is itself Lindelöf. If U is an open neighborhood of the intersection then $F_0 \setminus U$ is Lindelöf. The open cover of $F_0 \setminus U$ formed by relative complements of members of $\mathcal{F}$ with respect to F_0, has a countable subcover, say $(F_0 \setminus F_{\mu(i)})$, $i \in \mathbb{N}$. By the assumption of cofinality on ω_α, $\nu = \sup \mu(i)$ is less than ω_α and clearly from F_ν onwards members of the sequence $\mathcal{F}$ are contained in U.

[3] The referee has kindly pointed out that some of the statements (e.g., (a), (b) and (f)) can be found in a recent paper [5, Corollary 7.1, Theorems 1.5 and 7.6].

(f). On a Lindelöf, Hausdorff P-space (X, τ) let τ' be a topology larger than τ such that (X, τ') is also Lindelöf. Then unless $\tau = \tau'$, statement (c) (the continuous identity map from (X, τ') onto (X, τ) is a homeomorphism) provides a contradiction. Again, if (X, τ'') is a Hausdorff P-space such that τ'' is smaller than τ, then there is a closed hence Lindelöf subset A of (X, τ) which is not closed in (X, τ''). But τ'' being smaller than τ, A is a Lindelöf subset of (X, τ'') and this contradicts (a).

(g). Let $\mathcal{U}$ be any open cover of X. Then for each y in Y there is a countable subcover $\{U_i(y) | i \in \mathrm{N}\}$ of $f^{-1}(y)$ and also as f is closed, a neighborhood $N(y)$ of y, such that $f^{-1}[N(y)] \subset \bigcup_i U_i(y)$. Now, if Y is Lindelöf (respectively para-Lindelöf), the cover $\{N(y) | y \in Y\}$ has a countable subcover $\{N(y_i) | i \in \mathrm{N}\}$ (resp. a locally countable open refinement $\{M(y) | y \in Y\}$). It is easily seen that $\{U_i(y_j) | i, j \in \mathrm{N}\}$ (respectively $\{f^{-1}[M(y)] \cap U_i(y) | i \in \mathrm{N} \text{ and } y \in Y\}$) is a countable subcover (respectively locally-countable open refinement) of $\mathcal{U}$.

Other such preservation properties of P-proper maps can be similarly established.

Pursuing the same line, we give a few characterizations of the para-Lindelöf property (which, with no surprise, easily turns out to be equivalent to paracompactness) in regular P-spaces. These correspond to Michael's characterizations of paracompactness.

Theorem 4.3.[4] For a regular P-space X the following are equivalent:

(1) X is para-Lindelöf.

(2) Each open covering of X has a refinement $\bigcup_{\alpha < \omega_1} \mathcal{U}_\alpha$, where each $\mathcal{U}_\alpha$ is a locally countable family of open sets.

(3) Each open covering of X has a locally countable refinement.

(4) X is paracompact.

(5) Each open covering of X has a closed locally countable refinement.

Proof. (3) ⇒ (4) follows from the fact (contained in Lemma 4.4 below) that in a P-space every locally countable family is closure preserving and [16, Theorem 1]. The rest of the non-trivial implications have proofs parallel to those of the corresponding implications in [15; 16]. We shall just state two lemmas, without proof (see [8, pp. 82, 162]) and for a sample, prove only (2) ⇒ (3).

[4] The referee has also pointed out that in regular P-spaces the equivalence of paracompactness and the property of having a σ-locally countable open refinement for every open covering was proved by Frank Thall [19].

Lemma 4.4. If $\mathcal{D}$ is a locally countable family in a P-space, then
(i) $\overline{\mathcal{D}}$ is also locally countable;
(ii) for any subfamily $\mathcal{B}$ of $\mathcal{D}$, $\bigcup\overline{\mathcal{B}}$ is closed in X.

Lemma 4.5. Let $\mathcal{E}$ be a family of sets in a P-space X. Let $\mathcal{F}$ be a locally countable closed covering of X such that each of its members intersects only countably many members of $\mathcal{E}$. Then each member E of $\mathcal{E}$ can be enlarged to an open set $U(E)$ such that the family $\{U(E) \mid E \in \mathcal{E}\}$ is locally countable.

To prove (2) ⇒ (3), let $\mathcal{A}$ be an open covering of X. By (2) there is an open refinement $\bigcup_{\alpha<\omega_1} \mathcal{U}_\alpha$ where each $\mathcal{U}_\alpha$ is locally countable. For each α, let $V_\alpha = \bigcup \mathcal{U}_\alpha$. Then $\mathcal{V} = (V_\alpha)_{\alpha<\omega_1}$ is an open covering of X. For each $\beta < \omega_1$, let

$$W_\beta = V_\beta \setminus \bigcup_{\gamma<\beta} V_\gamma.$$

The family $(W_\alpha)_{\alpha<\omega_1}$ is a locally countable refinement of $\mathcal{V}$ and $\{W_\alpha \cap U \mid U \text{ in } \mathcal{U}_\alpha \text{ and } \alpha < \omega_1\}$ is a locally countable refinement of $\mathcal{A}$.

The above theorem together with known properties of paracompact spaces leads us to the following useful corollaries.

Corollary 4.6. Every regular $\aleph_1$-Lindelöf P-space is paracompact, hence normal.

Corollary 4.7. Every $F_{\sigma(1)}$-subspace of a regular paracompact P-space is paracompact.

Proof. Let $F = \bigcup \{F_\alpha \mid \alpha < \omega_1\}$ be an $F_{\sigma(1)}$-set in a paracompact P-space X and let $\mathcal{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ be an open cover of F. Each $U_\gamma = F \cap V_\gamma$, where V_γ is an open set in X. For each fixed $\alpha < \omega_1$, $\{V_\gamma \mid \gamma \in \Gamma\} \cup \{X \setminus V_\gamma\}$ is an open covering of X and so has an open locally finite refinement $\{W_{\gamma,\alpha} \mid \gamma \in \Gamma\}$. For each α, let $\mathcal{B}_\alpha = \{W_{\gamma,\alpha} \cap F \mid W_{\gamma,\alpha} \cap F_\alpha \neq \emptyset\}$; then $\mathcal{B}_\alpha$ is a locally finite family of open sets in F and $\bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$ is an open refinement of $\mathcal{U}$. Now application of Theorem 4.3 completes the proof.

We now consider the following anomalies:

Given a family of Hausdorff P-spaces, the usual product space is not a P-space but the P-product is. In terms of P-products the analogue of Tychonoff's theorem will read: The P-product of any family of Lindelöf P-spaces is Lindelöf. But this is not true. In fact the P-product of uncountably many copies of even the two point Hausdorff spaces is not Lindelöf.

This is due to the fact that every such space contains a closed copy of the countable P-product of the two point space which itself is an uncountable discrete space.

The lack of "P-productivity" for the Lindelöf property leads to another set back when one tries to find a maximal Lindelöf extension of a completely regular P-space on the lines of Stone-Čech compactification. To make it precise let a Lindelöf P-space Y be called a Lindelöf extension of a P-space X if there is a dense embedding of X into Y. A Lindelöf extension of X shall be called maximal if it maps continuously (preserving points of X) on every other Lindelöf extension of X. Now [12, Theorem 1] when applied to the case of P-spaces and Lindelöf extensions implies that every P-space which can be embedded in a product of Lindelöf P-spaces has a maximal Lindelöf extension if and only if the property of being a Lindelöf P-space is closed-hereditary and "P-productive". As has been shown above, the last mentioned condition is violated and hence the impossibility of getting maximal Lindelöf extensions in general.

In contrast to compactness, realcompactness allows a "P-product theorem" analogous to the theorem for Tychonoff products.

Theorem 4.8. An arbitrary P-product of realcompact P-spaces is realcompact.

Proof. Let $\{X_s \mid s \in S\}$ be a family of realcompact P-spaces and $Y = PX_s$ their P-product. Each projection $\mathrm{pr}_s: Y \to X_s$ has an extension $q_s: \upsilon Y \to X_s$, where υY is the Hewitt realcompactification of Y. Let a function $f: \upsilon Y \to Y$ be defined by $f(t) = (q_s(t))$. Then as $\mathrm{pr}_s \circ f = q_s$ is continuous, from Proposition [illegible].2, f is continuous. Now Y, being the set of fixed points of f, must be closed in υY and hence itself be realcompact. Consequently $\upsilon Y = Y$.

5. Connectedness of P-spaces

In general, P-spaces have poor connectedness properties. Completely regular P-spaces or equivalently regular P-spaces are all basically disconnected (see [10, Problem 4K]). Also, as shall be shown below, members of the larger class of functionally Hausdorff P-spaces are all totally disconnected.

It is a trivial proposition that a space is connected if and only if every continuous integer valued function on it is constant. For P-spaces the following is more interesting.

Proposition 5.1. A P-space X is connected if and only if every real-valued continuous function on X is a constant function. If a real-valued continuous function on X takes different values at two points of X then the two points lie in different components of X.

Proof As in a P-space zero-sets are precisely the clopen sets, a real-valued continuous function distinguishes two points of X if and only if they have disjoint clopen neighborhoods.

Corollary 5.2. A functionally Hausdorff P-space is totally disconnected.

Thus the search for connected P-spaces amounts to finding P-spaces on which all continuous real-valued functions are constant.

Every real-valued continuous function on the space E_0 of section 3.1 takes the same value at the two points a and b. We shall use this space and a condensation process [5] described below to construct a Hausdorff P-space E_ω on which every real-valued continuous function is constant.

Example 5.3 (The space E_ω). Let all the limit ordinals less than ω_1 be indexed by ω_1 in an order preserving manner. Let $i(\beta)$ denote the index of the limit ordinal β in the above indexing. First for the sake of clarity, let us consider an intuitive description of the space E_ω. From E_0 a space E_1 is constructed by attaching copies of the space E_0 to all pairs

$$(a_{\alpha\beta}, a_{\alpha,\beta+1}), (b_{\alpha\beta}, b_{\alpha,\beta+1}), (c_\beta, c_{\beta+1}), 0 < \alpha, \beta < \omega_1,$$

and

$$(a_{\alpha\gamma}, a_{\alpha,i(\gamma)}), (b_{\alpha\gamma}, b_{\alpha,i(\gamma)}), (c_\gamma, c_{i(\gamma)}), 0 \leqslant \alpha, \gamma < \omega_1$$

(where γ is a limit ordinal), of points in E_0. From E_1, a space E_2 is constructed by again similarly attaching new copies of E_0 to every newly attached copy of E_0 in E_1. This process is continued to give spaces E_n, $n \in \mathbb{N}$. Each E_n is embedded in E_{n+1} and the inductive limit of this sequence of spaces is the required space E_ω.

For a precise description let F_0 be the subspace $E_0 \setminus \{a, b\}$ of E_0 and consider the ordered pair (E_0, F_0). Suppose a pair (E_{k-1}, F_{k-1}) of spaces has been constructed, where F_{k-1} is a disjoint topological sum of an indexed collection $\mathcal{F}_{k-1}$ of copies of the space F_0 and is a subspace of E_{k-1}. We shall construct the next pair (E_k, F_k). For this, assume that P is the

[5] Other interesting examples obtained by using similar techniques can be found in [2;18;20].

indexing set of F_{k-1} and for any p in P the members of the corresponding copy of F_0 are distinguished by a superindex p. Also for simplification in the following description let d stand, in any particular pair, for any of the symbols a^p_α, b^p_α, c^p, $0 \leqslant \alpha < \omega_1$ and $p \in P$, such that whenever d stands for a^p_α or b^p_α, d_β means the point $a^p_{\alpha\beta}$ or $b^p_{\alpha\beta}$. Let the set of all symbols a^p_α, b^p_α, c^p for $0 \leqslant \alpha < \omega_1$ and p in P be denoted by S. We can form ordered pairs $(d_\beta, d_{\beta+1})$ and $(d_\gamma, d_{i(\gamma)})$, where in the latter pair γ is a limit ordinal. Let the collection of all such pairs for $d \in S$ and $0 \leqslant \beta, \gamma < \omega_1$, where γ is a limit ordinal, be denoted by P'. For every pair $(d_\beta, d_{\beta+1})$ or $(d_\gamma, d_{i(\gamma)})$ in P', let a copy $E_0(d_\beta, d_{\beta+1})$ or $E_0(d_\gamma, d_{i(\gamma)})$ of the space E_0 be taken and let G_k denote the disjoint topological sum of all the copies so obtained. When convenient, a pair in P' may be denoted by a single letter q and the points in the corresponding copy of E_0 can, as in the case of P, be distinguished by a superscript q. Let A be the subspace of G_k consisting of all points a^q and b^q, $q \in P'$. Clearly A is closed and discrete in G_k. Let a map $f\colon A \to F_{k-1}$ be defined, for all q in P', as follows:

$$\left.\begin{aligned} f(a^q) &= d_\beta \\ f(b^q) &= d_{\beta+1} \end{aligned}\right\} \quad \text{if } q = (d_\beta, d_{\beta+1}),$$

$$\left.\begin{aligned} f(a^q) &= d_\gamma \\ f(b^q) &= d_{i(\gamma)} \end{aligned}\right\} \quad \text{if } q = (d_\gamma, d_{i(\gamma)}),$$

Now let E_k be the adjunction space $G_k \cup_f E_{k-1}$. There is a natural embedding $\phi_{k-1}\colon E_{k-1} \to E_k$ by the definition of adjunction space (see [8, Theorem VI.6.3]). The space $G_k \backslash A$ is also embedded in E_k and is easily seen to be a disjoint topological sum of a collection of copies of the space F_0, the collection being indexed by P'. Let this collection be denoted by F_k and the space $E_k \backslash A$ by F_k. This completes the construction of the pair (E_k, F_k) and also by induction on k the construction of pairs (E_n, F_n) for every non-negative integer n.

Using maps $\phi_n\colon E_n \to E_{n+1}$, for each pair (m, n), $0 \leqslant m, n < \omega_0$, let a map $\phi^n_m\colon E_m \to E_n$ be defined by $\phi^n_m = \phi_{n-1} \circ \phi_{n-2} \circ \ldots \circ \phi_m$. These lead to an inductive system $\langle E_n, \phi^n_m \rangle$ of spaces and maps. The space E_ω is defined as the inductive limit of this system. The claimed properties of E_ω are stated in the following theorem.

Theorem 5.4. The space E_ω is a Hausdorff connected P-space.

Proof. That E_ω is a P-space easily follows from Proposition 2.1 since in the construction of E_ω only quotients and disjoint topological sums of P-spaces have been used. The space E_0 is Hausdorff and any continuous real-valued function on E_0 takes the same value at all points of $E_0 \backslash F_0$. Assume that in the pair (E_{k-1}, F_{k-1}), E_{k-1} is a Hausdorff space and every continuous real-valued function on E_{k-1} takes the same values at all points of $E_{k-1} \backslash F_{k-1}$. We shall prove the same for the next pair (E_k, F_k). By the very definition of E_k as the adjunction space $G_k \cup_f E_{k-1}$, it follows that E_{k-1} and F_k are embedded in E_k as disjoint closed and open subspaces respectively [8, Chapter 6]. Let x and y be any two points of E_k. For describing their disjoint neighborhoods, let it be agreed that by a *basic neighborhood* of a point in any copy of E_0, we shall mean the typical basic neighborhood defined in Example 3.1. Now if x and y are both in F_k, then disjoint neighborhoods of x and y in F_k are also disjoint neighborhoods in E_k. If x is in E_{k-1} and y in F_k, then a basic neighborhood of y in the copy of E_0 to which it belongs is in fact clopen in E_k. If both x and y are in E_{k-1}, let U_x and U_y be their open neighborhoods in E_{k-1}. For each point s of G_k which has been identified to some point in U_x take a basic neighborhood M_s of s in the copy of E_0 to which s belongs. Let V_x be the union of U_x with all such M_s's. The set V_x being a saturated (closed under the identification) open set in $G_k \oplus E_{k-1}$ gives a neighborhood U_x^+ of x in E_k. A neighborhood U_y^+ of y, similarly obtained is easily seen to be disjoint from U_x^+. This proves that E_k is Hausdorff. For the second assertion about the pair (E_k, F_k), we need simply observe that due to the chaining together of points by E_0 any closed neighborhood of a point in $E_k \backslash F_k$ always intersects $E_{k-1} \backslash F_{k-1}$. This together with the hypothesis on (E_{k-1}, F_{k-1}) and the fact that continuous real-valued functions on a P-space are locally constant proves that every continuous real-valued function on E_k takes the same value on $E_k \backslash F_k$.

Now given any two points in E_ω there always exists an integer n such that both the points lie in $E_n \backslash F_n$ where E_n is embedded in E_ω. Thus, every real-valued continuous function on E_ω is a constant function. To complete the proof we need only show that E_ω is a Hausdorff space. For this let x and y be two points in E_ω. There is a smallest integer n such that both x and y are in E_n. Let U_x and U_y be disjoint open neighborhoods of x and y in E_n. Earlier in this proof we have shown that U_x and U_y can be enlarged to disjoint open sets U_x^+ and U_y^+ in E_{n+1}. Thus for both x and y we get chains of relative neighborhoods. Due to the particular nature of the inductive system $\langle E_n, \phi_m^n \rangle$ where each ϕ_m^n is an embedding, the set E_ω is simply the union of the increasing sequence $\{E_n\}$ and the topology of E_ω is also the weak topology of the sum. Keeping this in mind it is easy to check that the sets obtained by taking unions of members in each chain of neighborhoods are disjoint open neighborhoods of x and y in E_ω. This completes the proof.

Closing this section it may be mentioned that an arbitrary P-product of connected P-spaces is connected. In fact for a fixed point in such a product the union of all connected subsets containing that point can be shown to be dense in the P-product.

6. Remarks

Non-P-spaces which admit only constant real-valued functions – there are even regular ones – show that even for a T_3 space X the condition 'every prime ideal in $C(X)$ is maximal' (see [10, Exercise 4J]) need not give a P-space. The quoted condition characterizes P-spaces among the Tychonoff spaces. A functionally Hausdorff P-space whose complete regularization (see [10, Theorem 3.9]) is not a P-space would prove the independence of the defining condition of a P-space from the above quoted condition. But presently we do not have an example of such a P-space. Comfort and Ross in [7] defined a P-space as a topological space in which every zero-set is open. This condition is certainly weaker than our defining condition. We do not have an example to show its independence of the prime-ideal condition above.

Acknowledgements

The author acknowledges his gratefulness to Professors S.P. Franklin and R.A. Alò of Carnegie-Mellon University for many useful suggestions and criticisms.

References

[1] Anderson, F.W., A class of function algebras, Canad. J. Math. 12 (1960) 353–366.

[2] Arhangel'skii, A.V. and S.P. Franklin, Ordinal invariants for topological spaces, Michigan Math. J. 15 (1968) 313–320.

[3] Bourbaki, N., Elements of mathematics. General Topology, part 1 (English transl.) (Addison-Wesley, Reading, Mass., 1966).

[4] Brainerd, B., F-rings of continuous functions, I, Canad. J. Math. 11 (1959) 80–86.

[5] Cameron, D.E., Maximal and minimal topologies, Trans. Amer. Math. Soc. 160 (1971) 229–248.
[6] Cohen, L.W. and C. Goffman, A theory of transfinite convergence, Trans. Amer. Math. Soc. 66 (1949) 65–74.
[7] Comfort, W.W. and K. Ross, On the finite product of topological spaces, Arch. Math. (Basel) 14 (1963) 62–63.
[8] Dugundji, J., Topology (Allyn and Bacon, Boston, Mass., 1966).
[9] Gillman, L. and M. Henriksen, Concerning rings of continuous functions, Trans. Amer. Math. Soc. 77 (1954) 340–362.
[10] Gillman, L. and M. Jerison, Rings of continuous functions (Van Nostrand, Princeton, N.J., 1960).
[11] Hanai, S., Inverse images of closed mappings, I, Proc. Japan. Acad. 37 (1961) 298–301.
[12] Herrlich, H. and J. van der Slot, Properties which are closely related to compactness, Indag. Math. 29 (5) (1967).
[13] Kelley, J.L., General topology (Van Nostrand, Princeton, N.J., 1955).
[14] Lorch, E.R., Compactifications, Baire functions and Daniel integration, Acta Sci. Math. Szeged 24 (1963) 204–218.
[15] Michael, E., A note on paracompact spaces, Proc. Amer. Math. Soc. 4 (1953) 831–838.
[16] Michael, E., Another note on paracompact spaces, Proc. Amer. Math. Soc. 8 (1957) 822–828.
[17] Sikorski, R., Remarks on spaces of high power, Fund. Math. 37 (1950) 125–136.
[18] Shimrat, M., Embedding in homogeneous spaces, Quart. J. Math. Oxford 5 (1954) 304–311.
[19] Tall, F., Thesis, Univ. of Wisconsin.
[20] Urysohn, P., Über die Mächtigkeit der zusammenhängenden Mengen, Ann. Math. 94 (1926) 262–295.